Performance Analysis of Space-Time MIMO Systems over Correlated Dynamic Channels

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Abstract—The closed-form simple expressions for bit error rate (BER) in multiple input – multiple output (MIMO) systems operating under flat and quasi-static Rayleigh fading have been derived. This method is based on the positional statistics. Next, the expressions were extended to dynamic and cross-correlated channels.

Index Terms - MIMO STC systems, fast fading, cross-correlation, positional statistics

I. INTRODUCTION

Potential performance benefits and a remarkable capacity promised by MIMO (multiple input – multiple output) systems attracted a lot of interest in the recent years. However, to keep this promise, many strict conditions have to be satisfied. This refers, inter alia, to the assumption of a quasi-static fading h(t)=h(t+T)= $\alpha\exp(-j\theta)$, where h - channel gain, α - module of a gain, θ - phase, T – signal symbol duration [1], [2]. Under real conditions, the function h(t) changes itself over the code word and this leads to interference [3], [5].

Analytical modeling of MIMO is very complex. For example, Jootar et al. [5] use the residue theorem to define the probability of error for the simplest 2-nd order scheme

$$P_b = \frac{1}{2\pi j}\int_{-\infty+je}^{\infty+je} \frac{1}{v\prod_1^4(1 - jv\lambda_i)}dv = -Res[\phi_z(s)/s \text{ at LHP poles }] \quad (1)$$

Alternatively, P_b can be found numerically by the Gauss-Chebyshev approximation [6] or via simulation [7].

In this paper the simple closed-form expressions are derived for a wide class of MIMO systems on the basis of positional statistics [8]. The proposed approach takes also into account the effect of movement and correlation.

The rest of the paper is organized as follows. Section II describes the system model. Section III deals with positional statistics and Selection Combining Diversity (SCD). In section IV the comparison of SCD and Maximum Ratio Combining (MRC) is carried out. Section V deals with cross-correlation and Section VI concludes the paper.

II. SYSTEM MODEL

The considered system consists of two transmitting and one, two or more receiving antennas, Fig 1. All antennas are omni-directional and potentially mobile up to the speed of several meters per second. The system exploits the space-time coding strategy [4]. In principle, the maximum likelihood decision rule is applied. The channel undergoes to flat (non-selective) Rayleigh fading. The channel gains are identical Gaussian random variables with zero means and autocorrelation function $1/2E[h_i(t)h_i^*(t+\tau)] = \sigma^2 R(l)$, where $R(l)$ follows Jakes' model [10]

$$R(l) = J_0(2\pi F_D T_s l) \quad (2)$$

$J_0(\cdot)$ is the zero-order Bessel function of the first kind (tabularized) and F_D stands for maximum Doppler shift and T_s is a symbol duration time, l = 1, 2, 3.. .

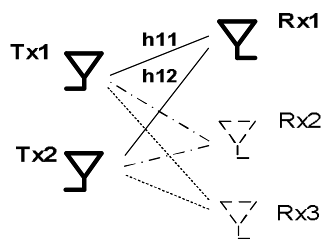

Figure 1. Considered MIMO system

The particular links between different antennas are identically distributed, while slightly cross-correlated. The perfect estimation of the functions *h(t)* is assumed.

For the 2x2 antennas system (MIMO2x2), the received signals in antenna Rx1 and Rx2 at some point of time *t* are r_{1t}, r_{2t}, respectively, while at the next moment *t+T* they are r_{1T}, r_{2T} (S_1,S_2 are useful signals from antennas 1,2; n_i is independent identically distributed complex Gaussian noise of zero mean and variance N_0)

$$\begin{aligned} r_{1t} &= h_{11t}S_1 + h_{12t}S_2 + n_1 \\ r_{2t} &= h_{21t}S_1 + h_{22t}S_2 + n_2 \\ r_{1T} &= h_{12T}S_1^* - h_{11T}S_2^* + n_3 \\ r_{2T} &= h_{22T}S_1^* - h_{21T}S_2^* + n_4 \end{aligned} \quad (3)$$

The changes in the third and fourth row of eq.(3) follow from the fact that signals S1, S2 in t+T period are conjugated and S1 is additionally multiplied by (-1).

Hence, eq. (3) can be transferred into a more conventional matrix form

$$\begin{bmatrix} r_{1t} \\ r_{1T}^* \\ r_{2t} \\ r_{2T}^* \end{bmatrix} = \begin{bmatrix} h_{11t} & h_{12t} \\ h_{12T}^* & -h_{11T}^* \\ h_{21t} & h_{22t} \\ h_{22T}^* & -h_{21T}^* \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2^* \\ n_3 \\ n_4^* \end{bmatrix} \Leftrightarrow \boldsymbol{r} = \boldsymbol{HS} + \boldsymbol{n} \quad (4)$$

The decoder proposed by Alamouti acts as follows

$$\widetilde{S} = H^H r \Rightarrow \widetilde{S} = H^H HS + H^H n \quad (5)$$

where $\widetilde{S}$ is an estimate of the transmitted symbol (S_1 or S_2). Excluding the noise component $H^H n$ in force of the assumption $h(t)=h(t+T)$, final formula takes the form

$$\begin{bmatrix} \widetilde{S}_1 \\ \widetilde{S}_2 \end{bmatrix} = \begin{bmatrix} |h_{11}|^2 + |h_{12}|^2 + |h_{21}|^2 + |h_{22}|^2 & 0 \\ 0 & |h_{11}|^2 + |h_{12}|^2 + |h_{21}|^2 + |h_{22}|^2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}. \quad (6)$$

In real conditions, however, the off-diagonal elements of $H^H H$ are no longer zero and they are the source of interference considered in this paper (section V).

III. SWITCHING DIVERSITY

Let us consider first the set of N independent channels, each described by the same kind of the density distribution function $f(\Gamma)$. Using this function one can arrange a new ordered set of channels, $m = 1, 2, \ldots, N$. The first channel denoted by $m=1$ is assigned by the lowest possible values of Γ, the next one of $m=2$ by the second lowest values and so on. The general formula for *m-th* channel is [8]

$$f_m(\Gamma \mid N) = \frac{N!\, f(\Gamma)}{(m-1)!\,(N-m)!} \cdot F^{m-1}(\Gamma)[1 - F(\Gamma)]^{N-m} \quad (7)$$

where $f(\Gamma)$, $F(\Gamma)$ are density and cumulative distribution functions (*ddf, cdf*), respectively.

Let Γ mean the signal-to-noise power ratio (SNR) in flat and slow Rayleigh fading. The *ddf* for a single channel is then as follows [9]

$$f(\gamma) = \gamma_0 e^{-\gamma/\gamma_0} \quad (8)$$

where γ_0 - mean value of the signal-to-noise power ratio. The *cdf* – being an integral of *ddf* - takes then a form

$$F(\gamma) = 1 - e^{-\gamma/\gamma_0} \quad (9)$$

Putting (8) and (9) into (7) one obtains

$$f_m(\gamma \mid N, \gamma_0) = \frac{N!}{(m-1)!\,(N-m)!} \cdot \gamma_0^{-1} e^{-\gamma/\gamma_0} \left(1 - e^{-\gamma/\gamma_0}\right)^{m-1} \left(e^{-\gamma/\gamma_0}\right)^{N-m} \quad (10)$$

From the viewpoint of bit error rate (BER) the most interesting is the sequence of $m = N$, which represents the virtual channel of the highest values of Γ. The equation (10) for $m = N$ takes the form

$$f_N(\gamma \mid \gamma_0) = \frac{N}{\gamma_0} e^{-\gamma/\gamma_0} (1 - e^{-\gamma/\gamma_0})^{N-1} \quad (11)$$

It represents the optimal channel in selection combining diversity.

Since the changes of γ are assumed slow, the resultant BER for the best channel can be calculated as the probability of error $P(\gamma)$ for no fading conditions averaged over the density distribution function (11)

$$P_N(\gamma_0) = \int_0^\infty P(\gamma) f_N(\gamma \mid \gamma_0) d\gamma \quad (12)$$

For the differential phase shift keying mode we have

$$P(\gamma) = \tfrac{1}{2} e^{-\gamma} \quad (13)$$

Putting (13) and (11) into (12) we finally obtain

$$P_N(\gamma_0) = \frac{N}{2\gamma_0} \int_0^\infty e^{-\gamma} e^{-\gamma/\gamma_0} (1 - e^{-\gamma/\gamma_0})^{N-1} d\gamma = \frac{N}{2} \sum_{k=0}^{N-1} (-1)^k \binom{N-1}{k} \frac{1}{\gamma_0 + k + 1} \quad (14)$$

The equation (14) defines the probability of error (BER) for MIMO systems with DPSK modulation and selection diversity combining. The variables are: the mean signal-to-noise power ratio γ_0 and the diversity order N. The BER curves for $N=1, 2, 4$, 6 are depicted in Fig. 2.

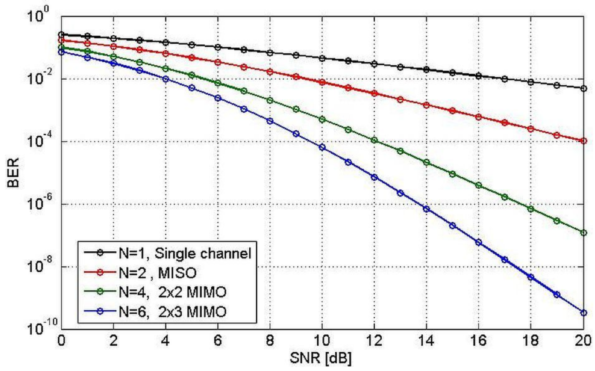

Figure 2. Original BER curves for N-th order switching diversity MIMO system with DPSK modulation

One can observe the regularity of the curves behavior and the fast decreasing BER with the increasing diversity order *N* and/or signal-to-noise power ratio SNR.

IV. SCD/MRC and DPSK/PSK Comparison

The selection diversity combining (SDC) exploits the best channel to detect the useful signal. The more often used, at least in the theory, is the maximum ratio combining (MRC), which exploits all the channels to detect this signal. Also, the coherent phase shift keying, PSK, is more often used as a reference model instead of DPSK. These differences may slightly affect the BER characteristics.

Let us consider first the consequence of the keying mode, PSK - DPSK. The known formula of Proakis for Rayleigh channel model and BPSK is [10]

$$P_{PP}(\gamma) = 0.5\left(1-\sqrt{\frac{\gamma}{\gamma+2}}\right) \tag{15}$$

It can be shown on the basis of series theory that this formula approaches eq. (14) for N=1 and for $\gamma\to\infty$ and $\gamma\to 0$. The in-between differences are as follows (Tab. I)

$$\Delta P_{PP} = \sqrt{\frac{\gamma}{\gamma+2}} - \frac{\gamma}{\gamma+1} \tag{16}$$

Table I
BER differences eq. (14, N=1) vs (15)

γ [W|dB]	0|-∞	1|0	10|10	100|20
ΔPPP [%]	0	8.47	0.286	0.0035

The formulae (14, N=1) and (15) are depicted in Fig.3. One can see that a difference in SNR is of the order of 0÷1 dB in favor of BPSK and it is observed mainly around the point $\gamma \approx 1$ (0 dB). Let us consider further the consequences of the switching/combining diversity.

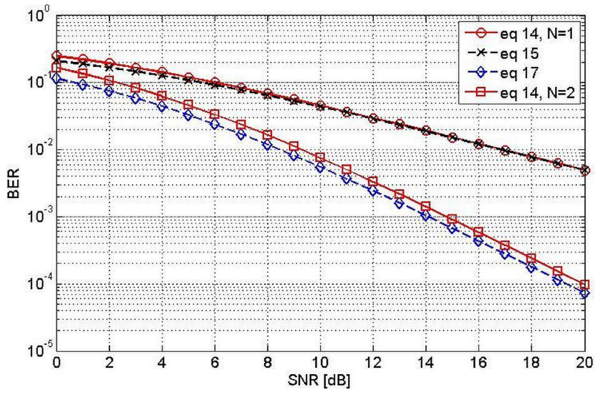

Figure 3. A comparison of formulae (14) with results of other authors

An approximation formula of Jootar et al. for MRC-BPSK system of N=2 is as follows [5]

$$P_{JP} \approx \frac{1}{4}\left(2+\sqrt{\frac{\gamma}{2\gamma+2}}\right)\left(1-\sqrt{\frac{\gamma}{2\gamma+2}}\right)^2 \tag{17}$$

The appropriate equation for switching diversity and DPSK is given by eq. (14) for N=2. Both formulae are depicted in Fig.2. One can see that a difference in SNR reaches 1÷2 dB in favor of MRC and BPSK.

The real channels usually are not as quasi-static as it was assumed before. Movement of receivers/ transmitters or other objects of real environment reveals as Doppler spread and can deteriorate performance. For MIMO with STBC (Space-Time Bloc Codes) this phenomenon leads to an irreducible error zone. The most popular way of investigation of this phenomenon is a simulation method. However, we propose some extensions to the previous formulas to include this phenomenon in eq. (14).

Let us consider the probability of error, which takes into account the Doppler effects according to [12]

$$P_{b_{floor}} = \frac{1}{2}\left[\frac{1+\gamma(1-\rho_C)}{1+\gamma}\right] \tag{18}$$

where ρ_C - autocorrelation function, eq. (2). It is evident that the error floor level depends on autocorrelation function.

To find the total probability we use similar approach as in [13]. Let us consider the interference – off-diagonal elements of eq. (6) - as white Gaussian noise. The signal-to-noise ratio in presence of interference is

$$\hat{\gamma} = \frac{\gamma \cdot \eta}{\gamma+\eta} \tag{19}$$

where η - signal-to-interference ratio.

$$\eta = \frac{1}{1 - J_0(2\pi F_D T_S)} \tag{20}$$

The term $F_D T_S$ is called normalized fading rate (NFR) or bandwidth and it expresses the grade of channel variability.

To achieve closed-form expressions for systems of N = 2, 4, 6.. formulae (17) has been taken as a reference. Then, the following correction is inserted to achieve the curves for MRC-BPSK system

$$\bar{\gamma} = 10^{n/10}\gamma \tag{21}$$

where n – number of receiving antennas.

Using (19) – (21) instead of γ in (14) one obtains the analytical expression for MIMO systems of N = 2, 4, 6.. . The comparison of these expressions and simulation results is presented in Fig. 4 and Fig. 5. Additionally, we modified (17) by means of (19) and (20). One can see that the agreement is fully satisfied.

V. Correlation

The PDF of two correlated identically distributed Rayleigh processes is given by [6]

$$p(\gamma) = \frac{1}{2\sqrt{\rho}\bar{\gamma}}\left[exp\left[-\frac{\gamma}{(1+\sqrt{\rho})\bar{\gamma}}\right] - exp\left[-\frac{\gamma}{(1-\sqrt{\rho})\bar{\gamma}}\right]\right] \quad \gamma \geq 0 \tag{22}$$

The probability of error for correlated quasi-static fading channel for N=2 and DPSK is as follows:

$$P_2 = \frac{1}{2}\int_0^\infty exp(-\gamma)\frac{1}{2\sqrt{\rho}\bar{\gamma}}\left[exp\left[-\frac{\gamma}{(1+\sqrt{\rho})\bar{\gamma}}\right] - exp\left[-\frac{\gamma}{(1-\sqrt{\rho})\bar{\gamma}}\right]\right]d\gamma \tag{23}$$

It should be pointed out, that coefficient ρ in (23) denotes the correlation coefficient of the Gaussian processes that produce the fading channels.

This coefficient is equal to the square root of the power correlation coefficient in Rayleigh fading.

$$\rho_G = \sqrt{\rho} \tag{24}$$

As a result the probability of error in correlated Rayleigh channels for MISO system is as follows

$$P_{2x1} = \frac{1}{2\left(\bar{\gamma}^2\left(1-\rho_G^2\right) + 2\bar{\gamma} + 1\right)} \tag{25}$$

Comparing (25) with (14) for N=2 we inserted an amendment to previous formulae to include correlation

$$\bar{\gamma}_{korr}^n = \bar{\gamma}_{iid}^n\left(1 - \bar{\rho}_G^2\right)^{n-1} \tag{26}$$

where $\bar{\rho}_G$ denotes mean of correlation coefficients. The results are depicted in Fig. 6 and Fig 7.

VI. Conclusion

The closed-form expressions for the bit error rate (BER) in switching diversity MIMO systems subjected to flat and quasi-static Rayleigh fading have been derived. This derivation is based on the positional statistics. Next the comparison of SCD and MRC was made. The obtained results were compared with the data of other authors. Then, the effects of movement and correlation on BER were taken into account. The proposed amendment, taking into account movement, can be also applied to extend formulae of other authors. Comparisons made with simulations data show satisfied agreement of results.

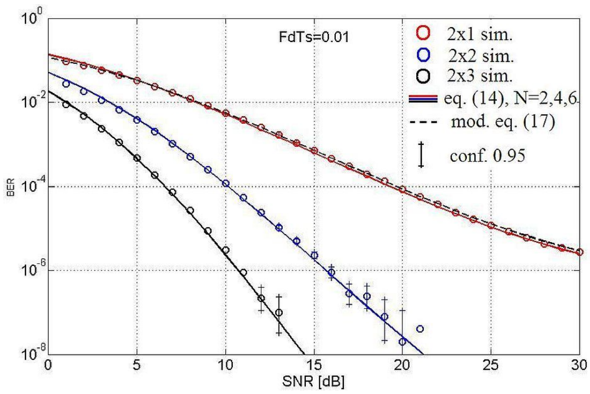

Figure 4. BER performance for quasi-static environment

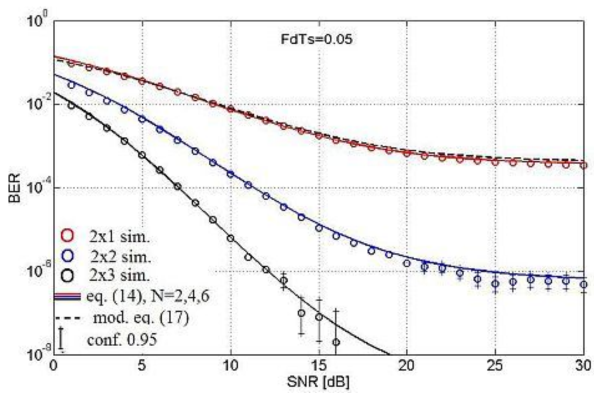

Figure 5. BER performance for moderate mobile conditions, $F_D T_S$ =0.05

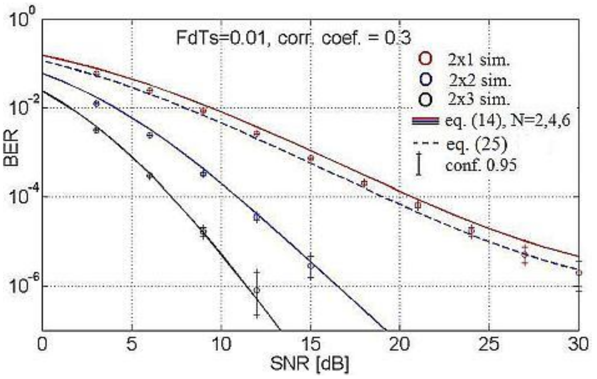

Figure 6. BER performance for quasi static and low corelated conditions

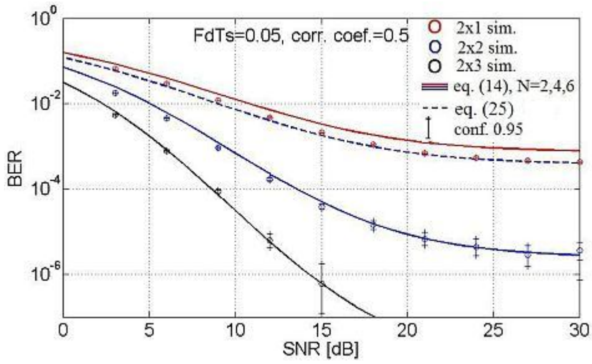

Figure 7. BER performance for moderate mobile and moderate cross-corelated conditions, F_DT_S=0.05, corr. coef = 0.5

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